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Determining optimal selling price and lot size with a varying rate of deterioration and exponential partial backlogging

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Abstract

In this paper, a deterministic inventory model for deteriorating items with price-dependent demand is developed. The demand and deterioration rates are continuous and differentiable function of price and time, respectively. In addition, we allow for shortages and the unsatisfied demand is partially backlogged at a negative exponential rate with the waiting time. Under these assumptions, for any given selling price, we first develop the criterion for the optimal solution for the replenishment schedule, and prove that the optimal replenishment policy not only exists but also is unique. If the criterion is not satisfied, the inventory system should not be operated. Next, we show that the total profit per unit time is a concave function of price when the replenishment schedule is given. We then provide a simple algorithm to find the optimal selling price and replenishment schedule for the proposed model. Finally, we use numerical examples to illustrate the algorithm.

Keywords: EOQ; Pricing; Deterioration; Partial backlogging

1. Introduction

In many inventory systems, the deterioration of goods is a realistic phenomenon. It is well known that certain products such as medicine, volatile liquids, blood bank, food stuff and many others, decrease under deterioration (vaporization, damage, spoilage, dryness and so on) during their normal storage period. As a result, while determining the optimal inventory policy of that type of products, the loss due to deterioration cannot be ignored. In the literature of inventory theory, the deteriorating inventory models have been continually modified so as to accommodate more practical features of the real inventory systems. The analysis of deteriorating inventory began with Ghare and Schrader [1], who established the classical no-shortage inventory model with a constant rate of decay. However, it has been empirically observed that failure and life expectancy of many items can be expressed in items of Weibull distribution. This empirical observation has prompted

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researchers to represent the time to deterioration of a product by a Weibull distribution. Covert and Philip [2] extended Ghare and Schrader's [1] model and obtain an economic order quantity model for a variable rate of deterioration by assuming a two-parameter Weibull distribution. Researchers including Philip [3], Misra [4], Tadikamalla [5], Wee [6], Chakrabarty et al. [7] and Mukhopadhyay et al. [8,9] developed economic order quantity models that focused on this type of products. Therefore, a realistic model is the one that treats the deterioration rate as a time varying function.

However, the above inventory models unrealistically assume that during stockout either all demand is backlogged or all is lost. In reality, often some customers are willing to wait until replenishment, especially if the wait will be short, while other are more impatient and go elsewhere. To reflect this phenomenon, Abad [10,11] discussed a pricing and lot-sizing problem for a product with a variable rate of deterioration, allowing shortages and partial backlogging. The backlogging rate depends on the time to replenishment – the longer customers must wait, the greater the fraction of lost sales. He presented two cases of the backlogging rates: $k_0 e^{-\delta x}$ and $\frac{k_0}{1+\delta x}$, where x is the waiting time up to the next replenishment, $0 < k_0 \leqslant 1$, and $\delta > 0$. However, he does not use the stockout cost (includes backorder cost and the lost sale cost) in the formulation of the objective function since these costs are not easy to estimate, and its immediate impact is that there is a lower service level to customers. Recently, Dye [12] amended Abad's [10,11] model by adding both the backorder cost and the cost of lost sales into the total profit. He considered the backlogging rate $\frac{1}{1+\delta x}$. The backlogging rate states that if customers do not have to wait, then no sales are lost, and all sales are lost if customers are faced with an infinite wait. However, the exponential backlogging rate remained unexplored.

The main purpose of this paper is to amend the paper of Abad [10,11] with a view to making the model more relevant and applicable in practice. We suppose that the fraction of customers who backlog their orders increases exponentially as the waiting time for the next replenishment decreases. The rest of the paper is organized as follows. In the next section, the assumptions and notation related to this study are presented. In Section 3, for any given selling price, we first develop the criterion for the optimal solution for the replenishment schedule, and prove that the optimal replenishment policy not only exists but also is unique. Next, we show that the total profit per unit time is a concave function of the selling price when the replenishment schedule is given. In the last two sections, numerical examples are discussed to illustrate the procedure of solving the model and concluding remarks are provided.

2. Notation and assumptions

2.1. Notation

To develop the mathematical model of inventory replenishment schedule, the notation adopted in this paper is as below:

A	the replenishment cost per order
c	the purchasing cost per unit
s	the selling price per unit, where $s > c$
Q	the ordering quantity per cycle
B	the maximum inventory level per cycle
c_1	the holding cost per unit per unit time
c_2	the backorder cost per unit per unit time
c_3	the cost of lost sales (i.e., goodwill cost) per unit
t_1	the time at which the inventory level reaches zero, $t_1 \geqslant 0$
t_2	the length of period during which shortages are allowed, $t_2 \geqslant 0$
T	the length of the inventory cycle, hence $T = t_1 + t_2$
$I_1(t)$	the level of positive inventory at time t, where $0 \leqslant t \leqslant t_1$
$I_2(t)$	the level of negative inventory at time t, where $t_1 \leqslant t \leqslant t_1 + t_2$
$\mathrm{TP}(s, t_1, t_2)$	the total profit per unit time

2.2. Assumptions

In addition, the following assumptions are imposed:

1. Replenishment rate is infinite, and lead time is zero.
2. The time horizon of the inventory system is infinite.
3. The demand rate, $d(s)$, is any non-negative, continuous, convex, decreasing function of the selling price in $[0, s_u]$.
4. The items deteriorate at a varying rate of deterioration $\theta(t)$, where $0 < \theta(t) \ll 1$.
5. Shortages are allowed. We adopt the concept used in Abad [10,11], where the unsatisfied demand is backlogged, and the fraction of shortages backordered is $e^{-\delta x}$, where x is the waiting time up to the next replenishment and δ is a positive constant.

2.3. Mathematical formulation

Using above assumptions, the inventory level follows the pattern depicted in Fig. 1. To establish the total relevant profit function, we consider the following time intervals separately, $[0, t_1]$ and $[t_1, t_1 + t_2]$. During the interval $[0, t_1]$, the inventory is depleted due to the combined effects of demand and deterioration. Hence the inventory level is governed by the following differential equation:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -d(s) - \theta(t)I_1(t), \quad 0 < t < t_1 \tag{1}$$

with the boundary condition $I_1(t_1) = 0$. Solving the differential Eq. (1), we get the inventory level as

$$I_1(t) = d(s)\mathrm{e}^{-g(t)} \int_t^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u, \quad 0 \leqslant t \leqslant t_1, \tag{2}$$

where $g(z) = \int_0^z \theta(u)\,\mathrm{d}u$.

Furthermore, at time t_1, shortage occurs and the inventory level starts dropping below 0. During the interval $[t_1, t_1 + t_2]$, the inventory level only depends on demand, and a fraction $\mathrm{e}^{-\delta(t_1+t_2-t)}$ of the demand is backlogged, where $t \in [t_1, t_1 + t_2]$. The inventory level is governed by the following differential equation:

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = -d(s)\mathrm{e}^{-\delta(t_1+t_2-t)}, \quad t_1 < t < t_1 + t_2 \tag{3}$$

with the boundary condition $I_2(t_1) = 0$. Solving the differential Eq. (3), we obtain the inventory level as

$$I_2(t) = -\frac{d(s)}{\delta}[\mathrm{e}^{-\delta(t_1+t_2-t)} - \mathrm{e}^{-\delta t_2}], \quad t_1 \leqslant t \leqslant t_1 + t_2. \tag{4}$$

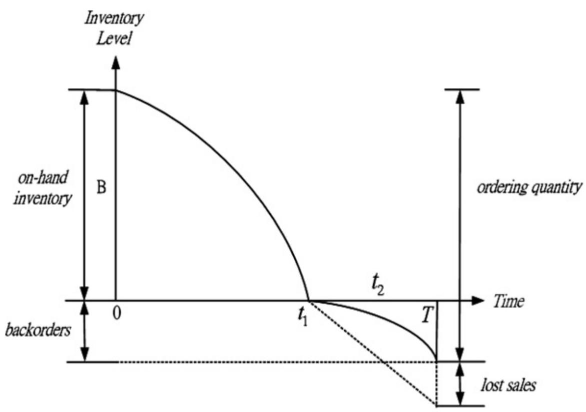

Fig. 1. Graphical representation of inventory system.

Therefore, the ordering quantity over the replenishment cycle can be determined as

$$Q = I_1(0) - I_2(t_1+t_2) = d(s)\left[\int_0^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u + \frac{1-\mathrm{e}^{-\delta t_2}}{\delta}\right] \tag{5}$$

and the maximum inventory level per cycle is

$$B = I_1(0) = d(s)\int_0^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u. \tag{6}$$

Based on Eqs. (2), (4) and (5), the total profit per cycle consists of the following elements:

1. ordering cost per cycle $= A$,
2. holding cost per cycle $= c_1\int_0^{t_1} I_1(t)\,\mathrm{d}t = c_1 d(s)\int_0^{t_1}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\mathrm{d}u\,\mathrm{d}t$,
3. backorder cost per cycle $= c_2\int_{t_1}^{t_1+t_2}[-I_2(t)]\mathrm{d}t = \frac{c_2 d(s)}{\delta^2}(1-\mathrm{e}^{-\delta t_2}-\delta t_2\mathrm{e}^{-\delta t_2})$,
4. opportunity cost due to lost sales per cycle $= c_3 d(s)\int_{t_1}^{t_1+t_2}[1-\mathrm{e}^{-\delta(t_1+t_2-t)}]\mathrm{d}t = \frac{c_3 d(s)}{\delta}(\mathrm{e}^{-\delta t_2}+\delta t_2-1)$,
5. purchase cost per cycle $= cQ = cd(s)\left[\int_0^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u + \frac{1-\mathrm{e}^{-\delta t_2}}{\delta}\right]$,
6. sales revenue per cycle $= s\left[\int_0^{t_1} d(s)\mathrm{d}u - I_2(t_1+t_2)\right] = sd(s)t_1 + \frac{sd(s)}{\delta}(1-\mathrm{e}^{-\delta t_2})$.

Therefore, the total profit per unit time of our model is obtained as follows:

$$\begin{aligned}
\mathrm{TP}(s,t_1,t_2) &= \frac{1}{t_1+t_2}\left\{\begin{array}{l}\text{sales revenue} - \text{ordering cost} - \text{holding cost}\\ -\text{backorder cost} - \text{opportunity cost} - \text{purchase cost}\end{array}\right\}\\
&= (s-c)d(s) - \frac{1}{t_1+t_2}\left\{\frac{[c_2-\delta(s-c+c_3)]d(s)}{\delta^2}(1-\delta t_2-\mathrm{e}^{-\delta t_2}) + A\right.\\
&\quad\left. + cd(s)\int_0^{t_1}[\mathrm{e}^{g(u)}-1]\mathrm{d}u + c_1 d(s)\int_0^{t_1}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t + \frac{c_2 d(s)}{\delta}t_2(1-\mathrm{e}^{-\delta t_2})\right\}.
\end{aligned} \tag{7}$$

To maximize the total profit per unit time, taking the first-order derivative of $\mathrm{TP}(s,t_1,t_2)$ with respect to t_1, t_2 and s, respectively, we obtain

$$\begin{aligned}
\frac{\partial \mathrm{TP}(s,t_1,t_2)}{\partial t_1} &= \frac{1}{(t_1+t_2)^2}\left\{\frac{[c_2-\delta(s-c+c_3)]d(s)}{\delta^2}(1-\delta t_2-\mathrm{e}^{-\delta t_2}) + A\right.\\
&\quad\left. + cd(s)\int_0^{t_1}[\mathrm{e}^{g(u)}-1]\mathrm{d}u + c_1 d(s)\int_0^{t_1}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\mathrm{d}u\,\mathrm{d}t + \frac{c_2 d(s)}{\delta}t_2(1-\mathrm{e}^{-\delta t_2})\right\}\\
&\quad - \frac{d(s)}{t_1+t_2}\left\{c[\mathrm{e}^{g(t_1)}-1] + c_1\int_0^{t_1}\mathrm{e}^{g(t_1)-g(t)}\mathrm{d}t\right\},
\end{aligned} \tag{8}$$

$$\begin{aligned}
\frac{\partial \mathrm{TP}(s,t_1,t_2)}{\partial t_2} &= \frac{1}{(t_1+t_2)^2}\left\{\frac{[c_2-\delta(s-c+c_3)]d(s)}{\delta^2}(1-\delta t_2-\mathrm{e}^{-\delta t_2}) + A\right.\\
&\quad\left. + cd(s)\int_0^{t_1}[\mathrm{e}^{g(u)}-1]\mathrm{d}u + c_1 d(s)\int_0^{t_1}\mathrm{e}^{-g(t)}\int_t^{t_1}\mathrm{e}^{g(u)}\mathrm{d}u\,\mathrm{d}t + \frac{c_2 d(s)}{\delta}t_2(1-\mathrm{e}^{-\delta t_2})\right\}\\
&\quad - \frac{d(s)}{t_1+t_2}[(s-c+c_3)(1-\mathrm{e}^{\delta t_2}) + c_2 t_2\mathrm{e}^{-\delta t_2}],
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\frac{\partial \mathrm{TP}(s,t_1,t_2)}{\partial s} &= [d(s)+(s-c)d'(s)]\left[1+\frac{1-\delta t_2-\mathrm{e}^{-\delta t_2}}{\delta(t_1+t_2)}\right]\\
&\quad - \frac{d'(s)}{t_1+t_2}\left\{\frac{c_2(1-\mathrm{e}^{-\delta t_2}-\delta t_2\mathrm{e}^{-\delta t_2})}{\delta^2} - \frac{c_3(1-\delta t_2-\mathrm{e}^{-\delta t_2})}{\delta}\right.\\
&\quad\left. + c\int_0^{t_1}[\mathrm{e}^{g(u)}-1]\mathrm{d}u + c_1\int_0^{t_1}\mathrm{e}^{-g(u)}\int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\right\}.
\end{aligned} \tag{10}$$

First, for any given selling price s, we want to prove that the optimal replenishment schedule not only exists but also is unique. To this end, the optimal solution of (t_1, t_2) must satisfy the equations $\frac{\partial \mathrm{TP}(t_1,t_2|s)}{\partial t_1} = 0$ and $\frac{\partial \mathrm{TP}(t_1,t_2|s)}{\partial t_2} = 0$ simultaneously, which implies

$$(s-c)d(s) - \mathrm{TP}(t_1,t_2|s) = d(s)\left\{c[\mathrm{e}^{g(t_1)} - 1] + c_1\int_0^{t_1} \mathrm{e}^{g(t_1)-g(t)}\,\mathrm{d}t\right\}, \tag{11}$$

and

$$(s-c)d(s) - \mathrm{TP}(t_1,t_2|s) = d(s)\left[(s-c+c_3)(1-\mathrm{e}^{-\delta t_2}) + c_2 t_2 \mathrm{e}^{-\delta t_2}\right], \tag{12}$$

respectively. Because both the left-hand sides in Eqs. (11) and (12) are the same, the right-hand sides in these equations are equal, that is,

$$(s-c+c_3)(1-\mathrm{e}^{-\delta t_2}) + c_2 t_2 \mathrm{e}^{-\delta t_2} = c[\mathrm{e}^{g(t_1)} - 1] + c_1\int_0^{t_1} \mathrm{e}^{g(t_1)-g(t)}\,\mathrm{d}t. \tag{13}$$

Furthermore, we substitute $\mathrm{TP}(t_1,t_2|s)$ in Eq. (7) into Eq. (12) and obtain

$$\begin{aligned} d(s)(t_1+t_2)\left[(s-c+c_3)(1-\mathrm{e}^{-\delta t_2}) + c_2 t_2 \mathrm{e}^{-\delta t_2}\right] = {} & \frac{[c_2 - \delta(s-c+c_3)]d(s)}{\delta^2} \times (1-\delta t_2 - \mathrm{e}^{-\delta t_2}) + A \\ & + cd(s)\int_0^{t_1}[\mathrm{e}^{g(u)} - 1]\mathrm{d}u + c_1 d(s)\int_0^{t_1} \mathrm{e}^{-g(t)} \\ & \times \int_t^{t_1} \mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t + \frac{c_2 d(s)}{\delta} t_2(1-\mathrm{e}^{-\delta t_2}). \end{aligned} \tag{14}$$

Taking the first-order derivative of the left-hand side of Eq. (13) with respect to t_2, it yields $[\delta(s-c+c_3) + c_2(1-\delta t_2)]\mathrm{e}^{-\delta t_2}$. Hence the left-hand side of Eq. (13) is a continuous function which increases strictly in $t_2 \in [0, \tilde{t}_2]$ and decreases strictly in $t_2 \in [\tilde{t}_2, \infty)$, respectively, where $\tilde{t}_2 = \frac{c_2 + \delta(s-c+c_3)}{c_2\delta}$. As a result, the left hand side of Eq. (13) has a maximum at the point $t_2 = \tilde{t}_2$, and is $s - c + c_3 + \frac{c_2}{\delta}\exp\left[-\frac{c_2+\delta(s-c+c_3)}{c_2}\right]$. On the other hand, because the right-hand side of Eq. (13) is a strictly increasing function of t_1 and it goes to infinite as $t_1 \to \infty$, there exists a unique $\tilde{t}_1$ such that

$$c\left[\mathrm{e}^{g(\tilde{t}_1)} - 1\right] + c_1\int_0^{\tilde{t}_1} \mathrm{e}^{g(\tilde{t}_1)-g(t)}\,\mathrm{d}t = s - c + c_3 + \frac{c_2}{\delta}\exp\left[-\frac{c_2 + \delta(s-c+c_2)}{c_2}\right].$$

Besides, for any given $t_2' \in (0, \tilde{t}_2)$, there exists a unique $t_1' \in (0, \tilde{t}_1)$ such that Eq. (13) holds. Similarly, for any given $t_2'' \in (\tilde{t}_2, \infty)$, we can also find a unique $t_1'' \in (0, \tilde{t}_1)$ such that Eq. (13) holds. Consequently, t_1 can be uniquely determined as a function of t_2.

Now, motivated by Eq. (14), we let

$$\begin{aligned} G(t_2) = {} & \frac{[c_2 - \delta(s-c+c_3)]d(s)}{\delta^2}(1-\delta t_2 - \mathrm{e}^{-\delta t_2}) + A + cd(s)\int_0^{t_1}[\mathrm{e}^{g(u)} - 1]\mathrm{d}u + c_1 d(s)\int_0^{t_1}\mathrm{e}^{-g(t)} \\ & \times \int_t^{t_1}\mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t + \frac{c_2 d(s)}{\delta} t_2(1-\mathrm{e}^{-\delta t_2}) - d(s)(t_1+t_2)\left[(s-c+c_3)(1-\mathrm{e}^{-\delta t_2}) + c_2 t_2\mathrm{e}^{-\delta t_2}\right]. \end{aligned} \tag{15}$$

After assembling Eq. (13), the first-order derivative of $G(t_2)$ with respect to t_2 becomes

$$\begin{aligned} \frac{\mathrm{d}G(t_2)}{\mathrm{d}t_2} &= d(s)\left\{c[\mathrm{e}^{g(t_1)} - 1] + c_1\int_0^{t_1}\mathrm{e}^{g(t_1)-g(t)}\,\mathrm{d}t\right\}\frac{\mathrm{d}t_1}{\mathrm{d}t_2} - d(s)[(s-c+c_3)(1-\mathrm{e}^{-\delta t_2}) + c_2 t_2\mathrm{e}^{-\delta t_2}]\frac{\mathrm{d}t_1}{\mathrm{d}t_2} \\ &\quad - d(s)(t_1+t_2)[\delta(s-c+c_3) + c_2(1-\delta t_2)]\mathrm{e}^{-\delta t_2} \\ &= -d(s)(t_1+t_2)[\delta(s-c+c_3) + c_2(1-\delta t_2)]\mathrm{e}^{-\delta t_2}. \end{aligned} \tag{16}$$

Then, we have the following result.

Theorem 1. *For any given s, we have*

(a) *If* $G(\tilde{t}_2) < 0$, *then the solution* (t_1^*, t_2^*) *which maximizes* $\mathrm{TP}(t_1, t_2|s)$ *not only exists but also is unique, and* $t_2^* \in (0, \tilde{t}_2)$.
(b) *If* $G(\tilde{t}_2) \geqslant 0$, *then the optimal value of* t_2 *is* $t_2^* \to \infty$.

Proof. See Appendix A for details. □

For any given positive s, Theorem 1(a) provides that if $G(\tilde{t}_2) < 0$, then we can find a unique point (t_1^*, t_2^*), where $t_2^* \in (0, \tilde{t}_2)$, such that the total profit per unit time $\mathrm{TP}(t_1^*, t_2^*|s)$ is maximum. Once the optimal solution (t_1^*, t_2^*) is obtained, the optimal $\mathrm{TP}(t_1^*, t_2^*|s)$ can be found from Eq. (12) and is

$$\mathrm{TP}(t_1^*, t_2^*|s) = (s-c)d(s) - d(s)\left[(s-c+c_3)(1-\mathrm{e}^{-\delta t_2^*}) + c_2 t_2^* \mathrm{e}^{-\delta t_2^*}\right].$$

On the other hand, Theorem 1(b) reveals that if $G(\tilde{t}_2) \geqslant 0$, then $t_2^* \to \infty$. By Eq. (13), the corresponding t_1^* is the point which satisfies $c[\mathrm{e}^{g(t_1^*)} - 1] + c_1 \int_0^{t_1^*} \mathrm{e}^{g(t_1^*)-g(t)}\,\mathrm{d}t = s - c + c_3$, and the optimal total profit per unit time is

$$\lim_{t_2^*\to\infty} \mathrm{TP}(t_1^*, t_2^*|s) = (s-c)d(s) - \lim_{t_2^*\to\infty} d(s)\left[(s-c+c_3)(1-\mathrm{e}^{-\delta t_2^*}) + c_2 t_2 \mathrm{e}^{-\delta t_2^*}\right] - \lim_{t_2^*\to\infty} \frac{G(t_2^*)}{t_1^* + t_2^*}. \tag{17}$$

Because $\lim_{t_2\to\infty}|G(t_2)| < \infty$, we have $\lim_{t_2^*\to\infty} \frac{G(t_2^*)}{t_1^*+t_2^*} = 0$. Hence Eq. (17) reduces to $\lim_{t_2^*\to\infty} \mathrm{TP}(t_1^*, t_2^*|s) = -c_3 d(s)$. The negative total profit per unit time, $-c_3 d(s)$, reveals that this given selling price is unsuitable. That is, at this given selling price s, the inventory system should not be operated. Once this case occurs, to improve the profit, we should rise the selling price.

Next, we study the condition under which the optimal selling price also exists and is unique. For given t_1^* and t_2^*, the first-order necessary condition for $\mathrm{TP}(s|t_1^*, t_2^*)$ to be maximum is

$$\begin{aligned}
\frac{\mathrm{d}\mathrm{TP}(s|t_1^*, t_2^*)}{\mathrm{d}s} &= [d(s) + (s-c)d'(s)]\left[1 + \frac{1-\delta t_2^* - \mathrm{e}^{-\delta t_2^*}}{\delta(t_1^*+t_2^*)}\right] \\
&\quad - \frac{d'(s)}{t_1^*+t_2^*}\left\{\frac{c_2\left(1-\mathrm{e}^{-\delta t_2^*} - \delta t_2^* \mathrm{e}^{-\delta t_2^*}\right)}{\delta^2} - \frac{c_3\left(1-\delta t_2^* - \mathrm{e}^{-\delta t_2^*}\right)}{\delta}\right. \\
&\quad \left. + c\int_0^{t_1^*}\left[\mathrm{e}^{g(u)} - 1\right]\mathrm{d}u + c_1 \int_0^{t_1^*} \mathrm{e}^{-g(t)} \int_t^{t_1^*} \mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\right\} \\
&= 0.
\end{aligned} \tag{18}$$

It is easy to see that $1 - \mathrm{e}^{-x} - x\mathrm{e}^{-x} > 0$ and $1 - x - \mathrm{e}^{-x} < 0$ for all $x > 0$. Hence we obtain $1 - \mathrm{e}^{-\delta t_2^*} - \delta t_2^* \mathrm{e}^{-\delta t_2^*} > 0$ and $1 - \delta t_2^* - \mathrm{e}^{-\delta t_2^*} < 0$. Besides, we have

$$1 + \frac{1-\delta t_2^* - \mathrm{e}^{-\delta t_2^*}}{\delta(t_1^*+t_2^*)} = \frac{t_1^*}{t_1^*+t_2^*} + \frac{1-\mathrm{e}^{-\delta t_2^*}}{\delta(t_1^*+t_2^*)} > 0.$$

According to the above, it is clear that the Eq. (18) has a solution only if $d(s) + (s-c)d'(s) < 0$. Further, if the gross profit is a strictly concave function of s, then $d(s) + (s-c)d'(s)$, which is the derivative of the concave $(s-c)d(s)$, is a strictly decreasing function of s, which implies $2d'(s) + (s-c)d''(s) < 0$, we then have

$$\begin{aligned}
\frac{d^2\mathrm{TP}(s|t_1^*, t_2^*)}{\mathrm{d}s^2} &= [2d'(s) + (s-c)d''(s)]\left[1 + \frac{1-\delta t_2^* - \mathrm{e}^{-\delta t_2^*}}{\delta(t_1^*+t_2^*)}\right] \\
&\quad - \frac{d''(s)}{t_1^*+t_2^*}\left\{\frac{c_2(1-\mathrm{e}^{-\delta t_2^*} - \delta t_2^* \mathrm{e}^{-\delta t_2^*})}{\delta^2} - \frac{c_3(1-\delta t_2^* - \mathrm{e}^{-\delta t_2^*})}{\delta} + c\int_0^{t_1^*}[\mathrm{e}^{g(u)} - 1]\mathrm{d}u\right. \\
&\quad \left. + c_1 \int_0^{t_1^*} \mathrm{e}^{-g(t)} \int_t^{t_1^*} \mathrm{e}^{g(u)}\,\mathrm{d}u\,\mathrm{d}t\right\} < 0.
\end{aligned}$$

Consequently, there exists a unique optimal selling price s^* which maximizes $\mathrm{TP}\left(s|t_1^*, t_2^*\right)$. The solution of $d(s) + (s - c)d'(s) = 0$, say s_l, is the lower bound for the optimal selling price s^* such that $\frac{\mathrm{dTP}\left(s|t_1^*, t_2^*\right)}{ds} = 0$.

Summarize the above results, we can now establish the following algorithm to obtain the optimal solution of our problem.

Algorithm

Step 1. Start with $j = 0$ and $s_j = s_l$, where s_l is a solution of $d(s) + (s - c)d'(s) = 0$.

Step 2. Put s_j and $\tilde{t}_2 = \frac{c_2+\delta(s_j-c+c_3)}{c_2\delta}$ into Eq. (13) to obtain the corresponding value of t_1, i.e., $\tilde{t}_1$, and then from Eq. (15) to calculate $G(\tilde{t}_2)$.

Step 3. If $G(\tilde{t}_2) < 0$, then go to Step 4. Otherwise, let $s_{j+1} = s_j + \varepsilon$, where ε is any small positive number, and set $j = j + 1$; then, go back to Step 2.

Step 4. From Eq. (15) to find the optimal value t_2^* such that $G(t_2^*) = 0$, and then put t_2^* into Eq. (13) to obtain the corresponding value of t_1, i.e., t_1^*, for a given selling price s_j.

Step 5. Use the result in Step 4 to determine the optimal s_{j+1} by Eq. (18).

Step 6. If the difference between s_j and s_{j+1} is sufficiently small, set $s^* = s_{j+1}$, then $\left(s^*, t_1^*, t_2^*\right)$ is the optimal solution and stop. Otherwise, set $j = j + 1$ and go back to Step 2.

3. Numerical example

To illustrate the results, let us apply the proposed algorithm to solve the following numerical examples.

Example 1. We first consider an inventory situation proposed by Wee [6]: $A = \$250$/per order, $c = \$8$/per unit, $c_1 = \$0.50$/per unit/per unit time, $c_2 = \$2.00$/per unit/per unit time, $d(s) = 25 - 0.5s$, where $s \in [0, 50]$, $\theta(t) = \alpha \times \beta \times t^{\beta-1} = 0.05 \times 1.5 \times t^{1.5-1} = 0.075t^{0.5}$ (e.g. Weibull deterioration rate, where α is scale parameter and β is shape parameter). Besides, we take $c_3 = \$2.00$/per unit and assume that the backlogging rate is $e^{-0.2x}$, where x is the waiting time up to the next replenishment. By solving $d(s) + (s - c)d'(s) = 0$, we obtain $s_l = s_0 = 29$. Then, applying the algorithm, the iterations to find the optimal replenishment policy are shown in Table 1. After five iterations, we have $s^* = 30.36569$, $t_1^* = 4.42898$, $t_2^* = 1.32528$. Therefore, from Eqs. (5) and (7), we obtain $Q^* = 64.3$ and $\mathrm{TP}^* = 143.91$.

The three-dimensional total profit per unit time graph as $s^* = 30.36569$ is shown in Fig. 2. Note that we run the numerical results with distinct starting values of $s = 15, 17.5, \ldots, 45$. The numerical results indicate that $\mathrm{TP}(s) = \mathrm{TP}(s|t_1^*, t_2^*)$ is strictly concave in s, as shown in Fig. 3. Consequently, we are sure that the local maximum obtained here indeed the global maximum solution.

Example 2. In this example, the same deterioration rate and backlogging rate in Example 1 are used. Then, we consider an inventory situation proposed by Dye [12]: $A = \$250$/per order, $c = \$40$/per unit, $c_1 = \$1.50$/per unit/per unit time, $c_2 = \$5.00$/per unit/per unit time, $c_3 = \$5.00$/per unit, $d(s) = 16 \times 10^7 \times s^{-3.21}$, where $s \in [0, 75]$. By solving $d(s) + (s - c)d'(s) = 0$, we obtain $s_l = s_0 = 58.0995$. Then, applying the algorithm, the iterations to find the optimal replenishment policy are shown in Table 2. After 4 iterations, we have $s^* = 59.19363$, $t_1^* = 0.59049$, $t_2^* = 0.18990$. Therefore, from Eqs. (5) and (7), we obtain $Q^* = 256.1$ and $\mathrm{TP}^* = 5690.02$.

The three-dimensional total profit per unit time graph as $s^* = 59.19363$ is shown in Fig. 4. Note that we run the numerical results with distinct starting values of $s = 45, 47.5, \ldots, 75$. The numerical results indicate that

Table 1
Iterations to find the optimal replenishment policy for Example 1

j	$G(\tilde{t}_2)$	t_1	t_2	s
1	−1793.996	4.31689	1.32286	30.31641
2	−1818.090	4.42479	1.32507	30.36382
3	−1818.679	4.42882	1.32527	30.36561
4	−1818.701	4.42897	1.32528	30.36568
5	−1818.702	4.42898	1.32528	30.36569

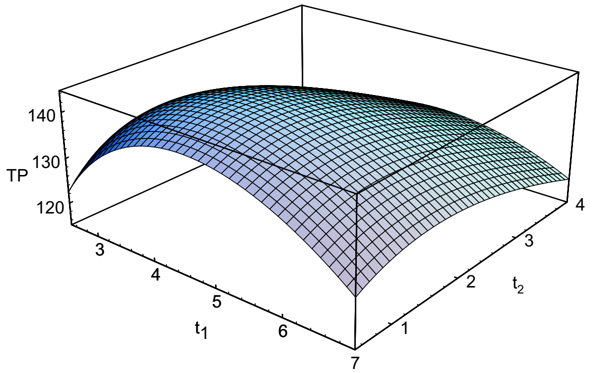

Fig. 2. The total profit per unit time, $T(t_1, t_2|s^* = 30.36569)$.

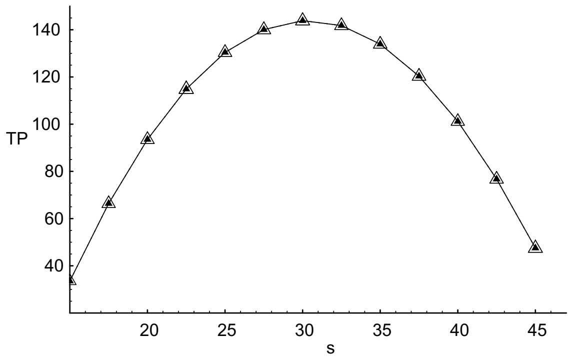

Fig. 3. Graphical representation of $\mathrm{TP}(s|t_1^*, t_2^*)$.

Table 2
Iterations to find the optimal replenishment policy for Example 2

j	$G(\tilde{t}_2)$	t_1	t_2	s
1	−43610.72	0.57442	0.18751	59.16020
2	−43332.12	0.59999	0.18983	59.19260
3	−43323.38	0.59048	0.18990	59.19360
4	−43323.10	0.59049	0.18990	59.19363

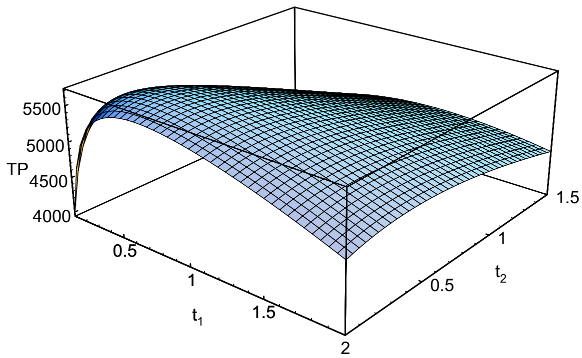

Fig. 4. The total profit per unit time, $\mathrm{TP}(t_1, t_2|s^* = 59.19363)$.

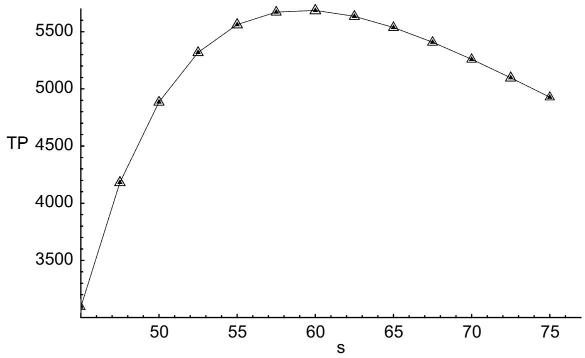

Fig. 5. Graphical representation of $\mathrm{TP}(s|t_1^*, t_2^*)$.

$\mathrm{TP}(s) = \mathrm{TP}(s|t_1^*, t_2^*)$ is strictly concave in s, as shown in Fig. 5. Consequently, we are sure that the local maximum obtained here is indeed the global maximum solution.

4. Concluding remarks

In this paper, we provide some useful properties for finding the optimal price and replenishment schedule under exponential partial backlogging. Since decision variables in our problem cannot be solved by simple algebraic means, they have to be solved numerically by using Newton–Raphson Method (or any bisection method). Based on our arguments, in order to find a value t_2^* such that $G(t_2^*) = 0$ and t_2^* is optimal, a proper choice of the initial value t_2 is very important due to possible local maxima. Without the right choice of initial value, for example, taking $t_2 > \tilde{t}_2$, Newton–Raphson Method fails to produce a solution satisfying the sufficient condition for the maximality problem of $\mathrm{TP}(t_1, t_2|s)$ and it will converge to a saddle point. In contrast to Wee [6] and Mukhopadhyay et al. [8,9], the approach in this paper provides solutions better than those obtained by using Taylor Series approximation. We can also see that any deterioration rate can be applied to this model such as the three-parameter Weibull deterioration rate (e.g., Philip [3]) and Gamma deterioration rate (e.g., Tadikamalla [5]). Hence the utilization of general price-dependent demand and deterioration rates make the scope of the application broader.

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Appendix A

The proof of Theorem 1

(a) Because we have

$$[\delta(s - c + c_3) + c_2(1 - \delta t_2)]\mathrm{e}^{-\delta t_2}\begin{cases} > 0, & \text{if } t_2 \in [0, \tilde{t}_2), \\ < 0, & \text{if } t_2 \in (\tilde{t}_2, \infty), \end{cases}$$

thus, from Eq. (16), we obtain

$$\frac{\mathrm{d}G(t_2)}{\mathrm{d}t_2}\begin{cases} < 0, & \text{if } t_2 \in [0, \tilde{t}_2), \\ > 0, & \text{if } t_2 \in (\tilde{t}_2, \infty), \end{cases}$$

which implies $G(t_2)$ is strictly decreasing in the interval $[0, \tilde{t}_2]$ and strictly increasing in the interval $[\tilde{t}_2, \infty)$, and has a minimum at point $t_2 = \tilde{t}_2$, i.e., $G(\tilde{t}_2)$ is the minimum value.

First, we consider the interval $[0, \tilde{t}_2]$. By using Eq. (13), we see that $t_1 = 0$ as $t_2 = 0$. Thus, we obtain $G(0) = A > 0$. Because $G(t_2)$ is strictly decreasing in the interval $[0, \tilde{t}_2]$, and on condition that $G(\tilde{t}_2) < 0$, from the Intermediate Value Theorem, we can find a unique solution $t_2^* \in (0, \tilde{t}_2)$ such that $G(t_2^*) = 0$.

Moreover, since $\delta(s - c + c_3) + c_2(1 - \delta t_2) > 0$ for $t_2 < \tilde{t}_2$, we then obtain

$$\begin{aligned}
\left.\frac{\partial^2 \mathrm{TP}(t_1, t_2|s)}{\partial t_2^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} &= \frac{1}{(t_1^* + t_2^*)^2}\Bigg\{-\frac{[c_2 - \delta(s - c + c_3)]d(s)}{\delta}(1 - \mathrm{e}^{-\delta t_2^*}) \\
&\quad + \frac{c_2 d(s)}{\delta}(1 - \mathrm{e}^{-\delta t_2^*} + \delta t_2^* \mathrm{e}^{-\delta t_2^*}) - d(s)[(s - c + c_3)(1 - \mathrm{e}^{-\delta t_2^*}) + c_2 t_2^* \mathrm{e}^{-\delta t_2^*}] \\
&\quad - d(s)(t_1^* + t_2^*)[\delta(s - c + c_3) + c_2(1 - \delta t_2^*)]\mathrm{e}^{-\delta t_2^*}\Bigg\} \\
&= \frac{-d(s)}{t_1^* + t_2^*}[\delta(s - c + c_3) + c_2(1 - \delta t_2^*)]\mathrm{e}^{-\delta t_2^*} < 0, \\
\left.\frac{\partial^2 \mathrm{TP}(t_1, t_2|s)}{\partial t_1^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} &= \frac{-d(s)}{t_1^* + t_2^*}\left\{c\theta(t_1^*)\mathrm{e}^{g(t_1^*)} + c_1\left[1 + \theta(t_1^*)\int_0^{t_1^*} \mathrm{e}^{g(t_1^*) - g(t)}\,\mathrm{d}t\right]\right\} < 0,
\end{aligned}$$

and

$$\left.\frac{\partial^2 \mathrm{TP}(t_1, t_2|s)}{\partial t_2 \partial t_1}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} = 0.$$

Thus, the determinant of the Hessian matrix at the stationary point (t_1^*, t_2^*) is

$$\begin{aligned}
\det(\mathbf{H}) &= \left[\frac{d(s)}{t_1^* + t_2^*}\right]^2 \left\{c\theta(t_1^*)\mathrm{e}^{g(t_1^*)} + c_1\left[1 + \theta(t_1^*)\int_0^{t_1^*} \mathrm{e}^{g(t_1^*) - g(t)}\,\mathrm{d}t\right]\right\} \\
&\quad \times \mathrm{e}^{-\delta t_2^*}[\delta(s - c + c_3) + c_2(1 - \delta t_2^*)] \\
&> 0.
\end{aligned}$$

As a result, we can conclude that the stationary point (t_1^*, t_2^*) is the optimal solution for our maximum problem.

Next, we consider the interval $[\tilde{t}_2, \infty)$. Because $G(t_2)$ is strictly increasing in the interval $[\tilde{t}_2, \infty)$, when $G(\tilde{t}_2) < 0$, which implies we cannot find a value $t_2 \in [\tilde{t}_2, \infty)$ such that $G(t_2) = 0$ (in this situation, the optimal solution of $t_2 \in [\tilde{t}_2, \infty)$ does not exist), or there exists a unique solution $t_2^{**} \in (\tilde{t}_2, \infty)$ such that $G(t_2^{**}) = 0$. For the latter situation, since $[\delta(s - c + c_3) + c_2(1 - \delta t_2)]\mathrm{e}^{-\delta t_2} < 0$ for $t_2 \in (\tilde{t}_2, \infty)$, the determinant of the Hessian matrix at the stationary point (t_1^{**}, t_2^{**}) is $\det(\mathbf{H}) < 0$. Consequently, (t_1^{**}, t_2^{**}) is not the optimal solution for our maximum problem. This completes the proof.

(b) Since $G(t_2)$ has a global minimum at $\tilde{t}_2$, if $G(\tilde{t}_2) > 0$, then we have $G(t_2) > G(\tilde{t}_2) > 0$ for all $t_2 \neq \tilde{t}_2$. Therefore, from Eqs. (9) and (15), we obtain that $\frac{\partial \mathrm{TP}(t_1, t_2, |s)}{\partial t_2} = \frac{G(t_2)}{(t_1 + t_2)^2} > 0$, which implies that a larger value of t_2 causes a higher value of $\mathrm{TP}(t_1, t_2|s)$. Hence the maximum value of $\mathrm{TP}(t_1, t_2|s)$ occurs at the point $t_2^* \to \infty$. For the another case $G(\tilde{t}_2) = 0$, since $\left.\frac{\partial \mathrm{TP}(t_1, t_2|s)}{\partial t_2}\right|_{t_2 = \tilde{t}_2} = 0$ and $\mathrm{TP}(t_1, t_2|s)$ is strictly increasing in $(0, \tilde{t}_2)$ and $(\tilde{t}_2, \infty)$, respectively. As a result, $t_2 = \tilde{t}_2$ is an inflection point and the maximum value of $\mathrm{TP}(t_1, t_2|s)$ occurs at the point $t_2^* \to \infty$. This completes the proof. □

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